IMPACT: International Journal of Research in Applied, Natural and Social Sciences (IMPACT: IJRANSS)
ISSN (P): 2347–4580; ISSN (E): 2321–8851
Vol. 10, Issue 8, Aug 2022, 49–56

PARAMETRIC METRIC SPACE AND FIXED POINT THEOREMS

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Received: 17 Aug 2022 **Accepted:** 22 Aug 2022 **Published:** 24 Aug 2022

ABSTRACT

There are many generalization of metric space. Parametric metric space is the generalization of metric space too. Which was introduced and studied by Hussian (a new approach to metric space) in 2014. In present paper we prove two fixed point theorems based on injective mapping using contraction conditions. Moreover, we provide an example to furnish our result and also the usability of our result.

Mathematics Subject Classification (MSC): *47H10, 54H25*

KEYWORDS: *Unique Fixed Point, Contraction, Parametric Metric Space, Injective Mapping*

INTRODUCTION

General Introduction

A metric on a nonempty set X is a mapping $d: X \times X \to [0. \infty]$ satisfying the following properties:

- $d(x, y) = 0$ If and if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(y, z) + d(z, y)$; then the pair (X, d) is said to be a metric spaces.

The theory of metric spaces is the general theory which covers several branches of mathematical analysis, as real analysis, complex analysis, multidimensional calculus, etc. Due to which, existence and uniqueness of fixed points and common fixed points has become a subject of great concern. In the recent six decades many authors generalized the Banach contraction Principle by moderating the triangular inequality of a metric space as generalized metric space[see 2,5,7 8,14,22 and references therein], cone metric space[see 9 references therein], b metric space[see 2,3,4,6 references therein], cone b metric space[see9,10,11,14-22 references therein], rectangular metric space [see 17 references therein], cone rectangular metric space [see 12,17,18 references therein], are some of the generalizations of metric space introduced by different authors in past few decades. Analogue Banach contraction principle, Kannan contraction principle, Ciric contraction principle and lots of the existing fixed point theorems for various generalized contractions were proved in these generalized spaces.

Most of the generalization of metric space are Hausdorff topology but we can also find generalization of metric space which are not necessarily Hausdorff topology (see, ref. [13, 19, 22,]). Tarskian mathematician used non Hausdorff topology for programming language semantics used in computer science.

The purpose of this paper is to prove some fixed point theorems for contraction mapping in parametric metric spaces an example is also given to distinguish our results.

PRELIMINARIES

Proceeding to our main result, let we furnish some definition , proposition, properties & lemmas needed in sequel.

1. Let X be a non empty set and Tp: $X \times X \times (0, \infty) \to (0, \infty)$ be a map on X such that $\forall\, x, y, z \in X$ and $t > 0$

- $Tp(x, y, t) = 0$ if and only if $x = y$
- $Tp(x, y, t) = Tp(y, x, t)$
- $Tp(x, y, t) \le Tp(x, z, t) + Tp(z, y, t)$

Then Tp is called parametric metric and pair (X, d) is called parametric metric space.

- If $\log_{n\to\infty}(x_n, x, t) = 0 \Rightarrow \log_{n\to\infty} x_n = x$, for all $t > 0$ then sequence $\{x_n\}_{n=1}^{\infty}$ converses $x \in X$
- If $\log_{n\to\infty}(x_n, x_m, t) = 0$ for all $t > 0$ then sequence $\{x_n\}_{n=1}^{\infty}$ is called Cauchy sequence.
- If every Cauchy sequence is convergent, then parametric metric space (X, d) is a complete parametric metric space.
- Let (X, d) be a parametric metric space and $T: X \to X$ be a mapping, then We say T is a continuous mapping at p in X, if for any sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that $\log_{n\to\infty} x_n = x \Rightarrow \log_{n\to\infty} Tx_n = Tx$.

Main Result

The objective of this paper is to prove some new fixed point theorems in parametric metric space. This paper is divided in two sections. In Section I and II we prove theorems on parametric metric spaces

SECTION I

Theorem 2

Let (X, d) be a complete parametric metric space and $Tp: X \to X$ be an injective mapping satisfying the condition

$$(2.1) \quad d(T_px, T_py, t) \le a.d(x, y, t) + b.d(x, T_px, t) + c.d(x, T_py, t) + d.\left(\frac{d(x,T_px,t)\,.d(y,T_py,t)}{d\{x,y\}+d(x,T_px,t)}\right) + e.\left(\frac{d(x,T_px,t).d(x,T_py,t)}{d\{x,y\}+d(x,T_px,t)}\right)$$

$\forall\, t \in [0,1)$; $a, b, c, d, e > 0$; $x, y \in X$ & $x \ne y$ have a fixed point if $a + b + 2c + d2 + e < 1$ and moreover a unique fixed point if $a + c < 1$.

Proof

Let $x_0 \in X$, Define iterative sequence$\{x_n\}_{n=1}^{\infty}$ follows: $Tpx_n = x_{n+1}$ for $n = 1, 2, 3, \ldots$. If for some n, $Tpx_n = x_n$, then x_n is the fixed point. Otherwise$Tpx_n \ne x_n$, using inequality (2.1)

$$d(x_{n+1},x_{n+2},t)=d(T_px_n,T_px_{n+1},t)$$
$$\leq a.d(x_n,x_{n+1},t)+b.d(x_n,T_px_n,t)+c.d(x_n,T_px_{n+1},t)$$
$$+d.\left(\frac{d(x_n,T_px_n,t).d(x_{n+1},T_px_{n+1},t)}{d(x_n,x_{n+1},t)+d(x_n,T_px_n,t)}\right)+e.\left(\frac{d(x_n,T_px_n,t).d(x_n,T_px_{n+1},t)}{d(x_n,x_{n+1},t)+d(x_n,x_{n+1},t)}\right)$$

$$d(x_{n+1},x_{n+2},t)$$
$$\leq a.d(x_n,x_{n+1},t)+b.d(x_n,x_{n+1},t)+c.d(x_n,x_{n+2},t)$$
$$+d.\left(\frac{d(x_n,x_{n+1},t).d(x_{n+1},x_{n+2},t)}{d(x_n,x_{n+1},t)+d(x_n,x_{n+1},t)}\right)+e.\left(\frac{d(x_n,x_{n+1},t).d(x_n,x_{n+2},t)}{d(x_n,x_{n+1},t)+d(x_n,x_{n+1},t)}\right)$$

$$d(x_{n+1},x_{n+2},t)$$
$$\leq a.d(x_n,x_{n+1},t)+b.d(x_n,x_{n+1},t)+c.[d(x_n,x_{n+1},t)+d(x_{n+1},x_{n+2},t)]$$
$$+d.\left(\frac{d(x_n,x_{n+1},t).d(x_{n+1},x_{n+2},t)}{2.d(x_n,x_{n+1},t)}\right)+e.\left(\frac{d(x_n,x_{n+1},t).d(x_n,x_{n+2},t)}{2.d(x_n,x_{n+1},t)}\right)$$

$$d(x_{n+1},x_{n+2},t)$$
$$\leq a.d(x_n,x_{n+1},t)+b.d(x_n,x_{n+1},t)+c.[d(x_n,x_{n+1},t)+d(x_{n+1},x_{n+2},t)]$$
$$+d.\left(\frac{d(x_n,x_{n+1},t).d(x_{n+1},x_{n+2},t)}{2.d(x_n,x_{n+1},t)}\right)+e.\left(\frac{d(x_n,x_{n+1},t).d(x_n,x_{n+2},t)}{2.d(x_n,x_{n+1},t)}\right)$$

$$d(x_{n+1},x_{n+2},t)$$
$$\leq a.d(x_n,x_{n+1},t)+b.d(x_n,x_{n+1},t)+c.[d(x_n,x_{n+1},t)+d(x_{n+1},x_{n+2},t)]$$
$$+d.\left(\frac{d(x_{n+1},x_{n+2},t)}{2}\right)+e.\left(\frac{d(x_n,x_{n+1},t)+d(x_{n+1},x_{n+2},t)}{2}\right)$$

$$\left(1-c-\frac{d}{2}-\frac{e}{2}\right)d(x_{n+1},x_{n+2},t)\leq\left(a+b+c+\frac{e}{2}\right)d(x_n,x_{n+1},t)$$

$$d\,(x_{n+1},x_{n+2},t)\leq\frac{\left(a+b+c+\frac{e}{2}\right)}{\left(1-c-\frac{d}{2}-\frac{e}{2}\right)}\,d(x_n,x_{n+1},t)$$

$$d\,(x_{n+1},x_{n+2},t)\leq k.d(x_n,x_{n+1},t)\forall\ t\ \in[0,1)\ and\ k=\frac{\left(a+b+c+\frac{e}{2}\right)}{\left(1-c-\frac{d}{2}-\frac{e}{2}\right)}<1\ \ \Rightarrow a+b+2c+\frac{d}{2}+$$

$e<1$.Therefore by successive iteration/ we have $d(xn+1, xn+2, t) \leq k^n\, d(x0, x1, t)$

As we know if $\{xn\}n\to\infty$ be a sequence in parametric space(X, d) such that $d(xn+1, xn+2, t) \leq k\ d(xn, xn+1, t)$ $\forall\ t \in [0,1)$ & $n = 1,2,3, \ldots$ then$\{xn\ \}n\to\infty$ is a Cauchy sequence in (X, d). Since (X, d) is a complete parametric space; $\{xn\}\ n\to\infty$ converses. Let $x* \in X$, then $limn\to\infty\ xn \to x*$. Again Tp is continuous, therefore

$$T_px^*=T_p(lim_{n\to\infty}x_n)=\ lim_{n\to\infty}T_px_n=x^*\Rightarrow T_px^*=x^*$$

Implies Tp has a fixed point $Tpx* = x*$ in X.

Now we will show that $x*$ is unique. For that, suppose $y*$ is another fixed point therefore$Tpy* = y*$. Therefore by inequality (2.1) we have

$$d(T_px^*, T_py^*, t)$$
$$\leq a.d(x^*, y^*, t) + b.d(x^*, T_px^*, t) + c.d(x^*, T_py^*, t)$$
$$+ d.\left(\frac{d(x^*T_px^*, t).d(y^*, T_py^*, t)}{d(x^*, y^*, t) + d(x^*, T_px^*, t)}\right) + e.\left(\frac{d(x^*, T_px^*, t).d(x^*, T_py^*, t)}{d(x^*, y^*, t) + d(x^*, T_px^*, t)}\right)$$

$$d(x^*, y^*, t) \leq a.d(x^*, y^*, t) + b.d(x^*, x^*, t) + c.d(x^*, y^*, t) + d.\left(\frac{d(x^*x^*, t).d(y^*, y^*, t)}{d(x^*, y^*, t) + d(x^*, x^*, t)}\right)$$
$$+ e.\left(\frac{d(x^*, x^*, t).d(x^*, y^*, t)}{d(x^*, y^*, t) + d(x^*, x^*, t)}\right)$$

$$d(x^*, y^*, t) \leq a.d(x^*, y^*, t) + c.d(x^*, y^*, t)$$
$$\Rightarrow (1 - a - c)d(x^*, y^*, t) \leq 0$$

$\Rightarrow d(x^*, y^*, t) = 0$ Since $a + c < 1 \Rightarrow x^* = y^*$. Hence T_p has a unique point.

SECTION II

Theorems

Let (X, Tp) be a complete parametric metric space and $Tp: X \rightarrow X$ be an injective mapping satisfying condition

$$(T_px, T_py, t) \leq \alpha\ Max\{d(x, y, t), \frac{d(x,T_px,t)d(y,T_py,t)}{d(x,y,t)}, \frac{d(x,T_py,t)d(y,T_px,t)}{d(x,y,t)}, \frac{d(x,T_px,t)d(x,T_py,t)}{2d(x,y,t)}\} (3.1)$$

$\forall\ t \in [0,1);\ \alpha > 0; x, y \in X\ \&\ x \neq y$ and $\alpha \in [0,1]$,then T_p has a unique fixed point.

Proof

Let $x0 \in X$ be an arbitrary point, Define iterative sequence $\{xn\}n=1\infty$ follows: $Tpxn = xn+1$for $n = 1, 2, 3, \ldots$. If for some n, $Tpxn = xn$, then xn is the fixed point. Otherwise $Tpxn \neq xn$, using inequality (3.1)

$$d(x_{n+1}, x_{n+2}, t) = d(T_px_n, T_px_{n+1}, t)$$
$$\leq \alpha\ Max\{d(x_n, x_{n+1}, t), \frac{d(x_n, T_p\,x_n, t)d(x_{n+1}, T_px_{n+1}, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, T_px_{n+1}, t)d(x_{n+1}, T_px_n, t)}{d(x_n, x_{n+1}, t)},$$
$$\frac{d(x_n, T_px_n, t)d(x_n, T_px_{n+1}, t)}{2d(x_n, x_{n+1}, t)}\}$$
$$\leq \alpha\ Max\{d(x_n, x_{n+1}, t), \frac{d(x_n, x_{n+1}, t)d(x_{n+1}, x_{n+2}, t)}{d(x_n, x_{n+1}, t)}, \frac{d(x_n, x_{n+2}, t)d(x_{n+1}, x_{n+1}, t)}{d(x_n, x_{n+1}, t)},$$

$$\frac{d(x_n,x_{n+1},t)d(x_n,x_{n+2},t)}{2d(x_n,x_{n+1},t)}\}$$

$$\leq \alpha\ Max\{d(x_n,x_{n+1},t),\frac{d(x_n,x_{n+1},t)d(x_{n+1},x_{n+2},t)}{d(x_n,x_{n+1},t)},0,\frac{d(x_n,x_{n+1},t)\{d(x_n,x_{n+1},t)+d(x_{n+1},x_{n+2},t)\}}{d(x_n,x_{n+1},t)}\}$$

$$\leq \alpha\ Max\{(d(x_n,x_{n+1},t),d(x_{n+1},x_{n+2},t),0,d(x_n,x_{n+1},t)+d(x_{n+1},x_{n+2},t)\}$$

$$\Rightarrow d(x_{n+1},x_{n+2},t)\leq \alpha\ d(x_n,x_{n+1},t)$$

Therefore by successive iteration

$$d(x_{n+1},x_{n+2},t)\leq \alpha^n\ d(x_0,x_1,t)$$

$$d(x_{n+1},x_{n+2},t)\leq \alpha^n\ d(x_0,x_1,t)$$

As we know if $\{xn\}n\to\infty$ be a sequence in parametric space(X, d) such that $d(xn+1, xn+2, t) \leq \alpha n\ d(x0, x1, t)$

$\forall\ t \in [0,1)$ & $n = 1,2,3, \ldots$ Then$\{xn\}n\to\infty$ is a Cauchy sequence in(X, d). Since (X, d) is a complete parametric space; $\{xn\}n\to\infty$ converses. Let $x* \in X$, then$limn\to\infty\ xn \to x*$. Again Tp is continuous, therefore

$$T_px^* = T_p(lim_{n\to\infty} x_n) = \ lim_{n\to\infty} T_px_n = x^* \Rightarrow T_px^* = x^*$$

Implies Tp has a fixed point $Tpx* = x*$ in X.

Now we will show that $x*$ is unique. for that suppose $y*$ is another fixed point therefore $Tpy* = y*$. Therefore by inequality (5.1) we have

$$d(T_px^*,T_py^*,t)$$
$$\leq \alpha\ Max\{d(x^*,y^*,t),\frac{d(x^*,T_px^*,t)d(y^*,T_py^*,t)}{d(x^*,y^*,t)},\frac{d(x^*,T_py^*,t)d(y^*,T_px^*,t)}{d(x^*,y^*,t)},\frac{d(x^*,T_px^*,t)d(x^*,T_py^*,t)}{2d(x^*,y^*,t)}\}$$

$$d(T_px^*,T_py^*,t)$$
$$\leq \alpha\ Max\{d(x^*,y^*,t),\frac{d(x^*,x^*,t)d(y^*,y^*,t)}{d(x^*,y^*,t)},\frac{d(x^*,y^*,t)d(y^*,x^*,t)}{d(x^*,y^*,t)},\frac{d(x^*,x^*,t)d(x^*,y^*,t)}{2d(x^*,y^*,t)}\}$$

$$d(T_px^*,T_py^*,t)\leq \alpha\ Max\{d(x^*,y^*,t),0,d(x^*,y^*,t),0\}$$

$$d(x^*,y^*,t)\leq \alpha d(x^*,y^*,t)$$

$$\Rightarrow (1-\alpha)d(x^*,y^*,t)\leq 0$$

$\Rightarrow d(x^*,y^*,t) = 0$ since $\alpha > 1 \Rightarrow x^* = y^*$. Hence T_p has a unique point.

Example: Let (X, d) be a complete parametric metric space, where $Tp: R+ \to R+$ is a mapping defined as $d(x, y, t) = t|x - y|q$ such that $x_n = 1 + \frac{1}{n}$ and $y_n = 1 + \frac{2}{n}$, therefore

$$d(x_n, y_n, t) = t|x_n - y_n|^q = t\left|1 + \frac{1}{n} - 1 - \frac{2}{n}\right|^q = t\left|-\frac{1}{n}\right|^q = t\,\frac{1}{n^q}$$

$$\log_{n\to\infty} d(x_n, y_n, t) = \log_{n\to\infty} t\,\frac{1}{n^q} = t\log_{n\to\infty}\frac{1}{n^q} = 0\ for\ t > 0$$

$$\Rightarrow \log_{n\to\infty} d(x_n, y_n, t) \to 0$$

as both $x_n = 1 + \frac{1}{n}$ and $y_n = 1 + \frac{2}{n}$ tends to 1 as $n \to \infty$. Hence 1 is the fixed point.

Hence it satisfies all the conditions of complete parametric metric space for $t > 0$ and of theorems [2,3].

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